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# A Unified Framework of Trade in Value Added; Physical, Monetary, Exchange Rates, and GHG Emissions

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**Abstract:** Koopman et al. (2014) developed a method to consistently decompose gross exports in value-added terms that accommodate infinite repercussions of international and inter-sector transactions. This provides a better understanding of trade in value added in global value chains than does the conventional gross exports method, which is affected by double-counting problems. However, the new framework is based on monetary input--output (IO) tables and cannot distinguish prices from quantities; thus, it is unable to consider financial adjustments through the exchange market. In this paper, we propose a framework based on a physical IO system, characterized by its linear programming equivalent that can clarify the various complexities relevant to the existing indicators and is proved to be consistent with Koopman's results when the physical decompositions are evaluated in monetary terms. While international monetary tables are typically described in current U.S. dollars, the physical framework can elucidate the impact of price adjustments through the exchange market. An iterative procedure to calculate the exchange rates is proposed, and we also show that the physical framework is also convenient for considering indicators associated with greenhouse gas (GHG) emissions.

**Keywords:** trade in value-added, global value chain, input-output
**JEL classification:** R15; C65; Q56

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# A Unified Framework of Trade in Value Added: Physical, Monetary, Exchange Rates, and GHG Emissions*

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February 22, 2016

**Abstract**

Koopman et al. (2014) developed a method to consistently decompose gross exports in value-added terms that accommodate infinite repercussions of international and inter-sector transactions. This provides a better understanding of trade in value added in global value chains than does the conventional gross exports method, which is affected by double-counting problems. However, the new framework is based on monetary input–output (IO) tables and cannot distinguish prices from quantities; thus, it is unable to consider financial adjustments through the exchange market. In this paper, we propose a framework based on a physical IO system, characterized by its linear programming equivalent that can clarify the various complexities relevant to the existing indicators and is proved to be consistent with Koopman's results when the physical decompositions are evaluated in monetary terms. While international monetary tables are typically described in current U.S. dollars, the physical framework can elucidate the impact of price adjustments through the exchange market. An iterative procedure to calculate the exchange rates is proposed, and some numerical exercises with hypothetical data are conducted to demonstrate the significance of local wages and capital flows, which are exogenous to the IO system. The physical framework is also convenient for considering indicators associated with greenhouse gas (GHG) emissions.

## 1 Introduction

The rise of global value chains (GVCs) during the last two decades has significantly changed the nature and structure of international trade, with many new implications for policymaking (Baldwin and Robert-Nicoud, 2014; Timmer et al., 2013). One of the most important features of GVCs is the transition of the trade pattern from "trade in goods" to "trade in tasks" (see Grossman and Rossi-Hansberg, 2008) in global production networks. This phenomenon has also been described as "the second great unbundling" (see Baldwin, 2012). The theoretical background is that the reduction of communication costs due to the IT revolution has enabled

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the international unbundling of factories and offices, which means that tasks can also be traded globally. In other words, countries no longer have to build or host the entire production chain; through the fragmentation of production, they can develop or attract productive capacity in a link of the chain where their comparative advantages fit the best. As a result, more and more intermediate goods, such as parts and components, are produced in sequential substages in different countries and then exported to other countries for use in further production. This, in turn, has significantly increased the complexity and sophistication of international production networks, bringing many new challenges in terms of how to better understand the creation, transfer and distribution of value added, income and job opportunities in GVCs.

Policy-makers require well-conceptualized indicators that can reveal the degree and nature of the interaction of their country with its major economic partners, the degree of GVC participation, and the location of their country in GVCs (see OECD-WTO-UNCTAD, 2013). Along these lines, many new indictors and measures based on input–output (IO) data have been proposed. Hummels et al. (2001) used the "import content of exports" indicator to measure a country's participation level in vertically specialized trade. Johnson and Noguera (2012) proposed the "trade in value added" (TiVA) indictor to measure how a country's value added is directly and indirectly absorbed by other countries' final demands through GVCs. Antràs et al. (2012) developed the concept of "distance," which is the number of stages that a product goes through before reaching the final demand, to measure the position of a country or industry in GVCs. Timmer et al. (2013) presented a new indicator for measuring the level of fragmentation of production. Koopman et al. (2014) developed a method to consistently decompose gross exports in value added terms, which provides a better understanding of value-added trade in GVCs as compared to the conventional gross exports, which is affected by double-counting problems. Wang et al. (2014) further extended the work of Koopman et al. to consistently measure value-added trade at bilateral and industrial levels.

However, the above efforts in developing the measurement of GVCs are all based on monetary IO tables, which cannot distinguish prices from quantities and are thus unable to consider financial adjustments through the exchange market. In this paper, we propose a more general framework based on a physical IO system to clarify the various complexities relevant to the existing IO-based GVCs indicators. Since the international monetary IO tables are generally described in current U.S. dollars, the physical framework can elucidate the impact of price adjustments through the exchange market. An iterative procedure to calculate the exchange rates is also proposed, and some numerical exercises with hypothetical data are conducted to demonstrate the significance of local wages and capital flows, which are exogenous to the IO system. The physical framework proposed is also convenient for considering the indicators associated with greenhouse gas (GHG) emissions.

In the following, after reviewing the linear programming problem of the one-country physical IO system, the problem for a world comprising two countries is formulated. With the physical system, it is easy to calculate the contributions of individual sectors and countries to each commodity price after considering the infinite number of repercussions of intermediate trade. Once such contributions are evaluated, it is easy to decompose the GDP of each country, which is the sum of values added. In section 4, the system is generalized to include $n$ sectors and $m$ countries. In section 5, the correspondence between physical and monetary systems is discussed to show that our results are essentially the same as the ones of Koopman et al. Returning to the physical system, the iterative process to endogenize the exchange rates is discussed in section 6. However, it presupposes the existence of outside systems to determine wage levels and capital flows, such as the labor and international financial markets. Greenhouse gases can also be incorporated in the commodities traded. A similar approach can then be used to determine who is ultimately responsible for emissions after taking into account all repercussions, which is the subject to be discussed in sections 7 and 8.

## 2 One-country physical table

The linear programming problem proposed by Dorfman et al. (1958) is convenient to formalize the physical IO system, and it might be beneficial to review the single country case as a starting point. The problem is to find the output schedule $x$ that minimizes the labor cost needed to satisfy the final demand requirement $y$:

$$\min_{x}\{wa_0x|(I-A)x \geq y, x \geq 0\}, \tag{1}$$

where $w$, $a_0$, and $A$, respectively, are the prevailing wage, labor (value added) input coefficient vector, and input coefficient matrix.

The Lagrangian function for the problem can be written using a row vector of multipliers $p$ as

$$L = wa_0x + p(y - (I-A)x). \tag{2}$$

Then one of the first-order conditions is

$$\frac{\partial L}{\partial x} = wa_0 - p(I-A) \geq 0, \tag{3}$$

where $p$ can be interpreted as the price vector.

When $p$ is positive, then the usual output equation is obtained as the optimal solution, viz., $x = (I-A)^{-1}y$. Conversely, when the output vector $x$ is positive, the row vector of commodity prices can be determined:

$$p = wa_0(I-A)^{-1}, \tag{4}$$

which is positive when the wage is positive, the labor inputs are non-negative but non-zero, and the Leontief inverse is positive definite.

Equation (4) can be decomposed as the sum of an infinite geometric series,

$$p = wa_0 + wa_0A + wa_0A^2 + wa_0A^3 + \cdots.$$

The first term represents the direct labor cost included in the product price while the second term represents the first-round repercussion as intermediate inputs to another commodity, and so forth.

When $b_{ij}$ denotes the $(i,j)$ element of the Leontief inverse, the price of commodity $i$ can be written as a weighted sum of the labor costs in all the sectors: $p_i = w\sum_j a_{0j}b_{ji}$. Then the portion of the price of commodity $i$ attributable to commodity $j$ as an intermediate input can be calculated as

$$c_{ji} = \frac{wa_{oj}b_{ji}}{p_i}. \tag{5}$$

# 3 Two-country physical table

A similar analysis applies to the case when commodity price composition in terms of origins of intermediate inputs is considered. In this case, country 1's problem is to minimize the costs of labor and imported intermediate inputs required to produce the domestic outputs,

$$\min_{x^1}\{(w^1 a_0^1 + p^2 A^{21})x^1 | (I - A^{11})x^1 - A^{12}x^2 \geq y^{11} + y^{12}, x^1 \geq 0\}, \tag{6}$$

and the problem of country 2 would be

$$\min_{x^2}\{(w^2 a_0^2 + p^1 A^{12})x^2 | - A^{21}x^1 + (I - A^{22})x^2 \geq y^{21} + y^{22}, x^2 \geq 0\}, \tag{6$'$}$$

where the superscripts 1 and 2 indicate the respective countries, $y^{rs}$ represents the amounts of country $r$'s products consumed as the final demand in country $s$, and $A^{rs}$ denotes the submatrix of interregional input coefficient matrix. Each country regards the price of imports $p^s$ as well as the domestic wage $w^r$ as being exogenous.

As each country regards the outputs of the other country as being exogenous, the problem can be described as a Nash problem, and $x^1$ and $x^2$ at the equilibrium are determined by solving (6) and (6') simultaneously. The same output schedule can be obtained from the *world problem* combining both countries.

$$\begin{array}{ll} \min_{x^1,x^2} & w^1 a_0^1 x^1 + w^2 a_0^2 x^2, \\ \text{s.t.} & (I - A^{11})x^1 - A^{12}x^2 \geq y^{11} + y^{12}, \\ & -A^{21}x^1 + (I - A^{22})x^2 \geq y^{21} + y^{22}, \\ & x^1 \geq 0, \quad \text{and} \quad x^2 \geq 0, \end{array} \tag{7}$$

and the Lagrangian function for the problem can be written with the multipliers $p^1$ and $p^2$ for the respective countries:

$$L = w^1 a_0^1 x^1 + w^2 a_0^2 x^2 + p^1(y^{11} + y^{12} - (I - A^{11})x^1 + A^{12}x^2) + p^2(y^{21} + y^{22} + A^{21}x^1 - (I - A^{22})x^2). \tag{8}$$

Two of the first-order conditions are

$$\begin{aligned} \frac{\partial L}{\partial x^1} &= w^1 a_0^1 - p^1(I - A^{11}) + p^2 A^{21} \geq 0 \\ \frac{\partial L}{\partial x^2} &= w^2 a_0^2 + p^1 A^{12} - p^2(I - A^{22}) \geq 0 \end{aligned} \tag{9}$$

If the output vectors are positive, then the price vectors can be found in matrix form:

$$(p^1 \quad p^2) = (w^1 a_0^1 \quad w^2 a_0^2) \begin{pmatrix} I - A^{11} & -A^{12} \\ -A^{21} & I - A^{22} \end{pmatrix}^{-1} = (w^1 a_0^1 \quad w^2 a_0^2) \begin{pmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{pmatrix} \tag{10}$$

Denoting the transaction between sectors $i$ and $j$ in the submatrix $B^{rs}$ by $b_{ij}^{rs}$, the price of commodity $i$ produced in country 1 can be decomposed as

$$p_i^1 = w^1 \sum_j a_{0j}^1 b_{ji}^{11} + w^2 \sum_j a_{0j}^2 b_{ji}^{21}. \tag{11}$$

Then, in the two-country framework, the portion of the prices of country 1's product that is attributed to value added originating in countries 1 and 2 can easily be calculated:

$$c_i^{11} = \frac{w^1 \sum_j a_{0j}^1 b_{ji}^{11}}{p_i^1} \quad \text{and} \quad c_i^{21} = \frac{w^2 \sum_j a_{0j}^2 b_{ji}^{21}}{p_i^1}.$$

Note that these expressions can directly be extended to the multi-country case. Since the values added can be attributed to each industry in each region, the portions of commodity $i$'s price produced in country $s$ that is attributable to industry $j$ in country $r$, and their aggregation by the originating country can be written, respectively, as

$$c_{ji}^{rs} = \frac{w^r a_{0j}^r b_{ji}^{rs}}{p_i^s} \quad \text{and} \quad c_i^{rs} = \frac{w^r \sum_j a_{0j}^r b_{ji}^{rs}}{p_i^s} \tag{12}$$

# 4 Decomposition of GDPs

By definition, the GDP of country 1 is given by

$$\begin{aligned} Y^1 &= p^1 y^{11} + p^2 y^{21} + p^1(y^{12} + A^{12}x^2) - p^2(y^{21} + A^{21}x^1) \\ &= \sum_i p_i^1 (y_i^{11} + y_i^{12} + \sum_j a_{ij}^{12} x_j^2) - \sum_i p_i^2 \sum_j a_{ij}^{21} x_j^1, \end{aligned} \tag{13}$$

where the first line represents the final demand for the domestic product $p^1 y^{11}$ plus the exports minus the imports. The exports and imports include both final and intermediate demands, and are evaluated using the prices of their origin. Likewise the GDP of country 2 can be written as

$$\begin{aligned} Y^2 &= p^2 y^{22} + p^1 y^{12} + p^2(y^{21} + A^{21}x^1) - p^1(y^{12} + A^{12}x^2) \\ &= \sum_i p_i^2 (y_i^{22} + y_i^{21} + \sum_j a_{ij}^{21} x_j^1) - \sum_i p_i^1 \sum_j a_{ij}^{12} x_j^2. \end{aligned}$$

By using the portions $c_i^{rs}$ defined in (12), these GDPs can be decomposed into the contributions of the respective countries, taking into account all the repercussions of intermediate transactions. That is, $Y^1 = Y^{11} + Y^{21}$ and $Y^2 = Y^{22} + Y^{12}$, where $Y^{rs}$ represents the part of country $s$'s GDP that is eventually attributable to country $r$.

$$\begin{aligned} Y^{11} &= \sum_i c_i^{11} p_i^1 (y_i^{11} + y_i^{12} + \sum_j a_{ij}^{12} x_j^2) - \sum_i (1 - c_i^{12}) p_i^2 \sum_j a_{ij}^{21} x_j^1, \\ Y^{21} &= \sum_i c_i^{21} p_i^1 (y_i^{11} + y_i^{12} + \sum_j a_{ij}^{12} x_j^2) - \sum_i c_i^{12} p_i^2 \sum_j a_{ij}^{21} x_j^1, \\ Y^{22} &= \sum_i c_i^{22} p_i^2 (y_i^{22} + y_i^{21} + \sum_j a_{ij}^{21} x_j^1) - \sum_i (1 - c_i^{21}) p_i^1 \sum_j a_{ij}^{12} x_j^2, \\ Y^{12} &= \sum_i c_i^{12} p_i^2 (y_i^{22} + y_i^{21} + \sum_j a_{ij}^{21} x_j^1) - \sum_i c_i^{21} p_i^1 \sum_j a_{ij}^{12} x_j^2. \end{aligned} \tag{14}$$

The second terms in the right hand sides of the above represent the imports. In the case of $Y^{11}$, only the share $c^{22}$ is subtracted because the remainder, $c^{12}$, is the portion of prices attributable to the economy of country 1, which does not need to be subtracted. Likewise, the second term of $Y^{21}$ subtracts country 1's contribution from the imports from country 2, since that part must be attributed to country 1 rather than country 2.[1]

Suppose there are $i, j = 1, \ldots, n$ commodities and $r, s = 1, \ldots, m$ countries, and that $p^s$ denotes the $(1 \times n)$ vector of FOB prices in country $s$. Further, introduce a diagonal matrix $C^{rs}$ comprising $\{c_1^{rs}, \ldots, c_n^{rs}\}$ obtained in (12). Then the general formulae of the GDP decompositions can be written as

$$\begin{aligned} Y^{ss} &= p^s C^{ss}(\sum_r y^{sr} + \sum_{r \neq s} A^{sr} x^r) - \sum_{r \neq s} p^r (I_n - C^{sr}) A^{rs} x^s \\ Y^{rs} &= p^s C^{rs}(\sum_r y^{sr} + \sum_{r \neq s} A^{sr} x^r) - p^r C^{sr} A^{rs} x^s, \quad (r \neq s). \end{aligned} \tag{15}$$

It is difficult to describe (15) in a simple matrix expression. For example, when there are three countries, the representation below provides one such expression.

$$\begin{aligned} &\begin{pmatrix} Y^{11} & Y^{12} & Y^{13} \\ Y^{21} & Y^{22} & Y^{23} \\ Y^{31} & Y^{32} & Y^{33} \end{pmatrix} \\ &= P^0 \begin{pmatrix} C^{11} & C^{12} & C^{13} \\ C^{21} & C^{22} & C^{23} \\ C^{31} & C^{32} & C^{33} \end{pmatrix} \begin{pmatrix} \sum_r y^{1r} + \sum_{r \neq 1} A^{1r} x^r & 0 & 0 \\ 0 & \sum_r y^{2r} + \sum_{r \neq 2} A^{2r} x^r & 0 \\ 0 & 0 & \sum_r y^{3r} + \sum_{r \neq 3} A^{3r} x^r \end{pmatrix} \\ &- \begin{pmatrix} 0 & p^2(I - C^{12}) & p^3(I - C^{13}) \\ 0 & p^2 C^{12} & 0 \\ 0 & 0 & p^3 C^{13} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ A^{21} x^1 & 0 & 0 \\ A^{31} x^1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} p^1 C^{21} & 0 & 0 \\ p^1(I - C^{21}) & 0 & p^3(I - C^{23}) \\ 0 & 0 & p^3 C^{23} \end{pmatrix} \begin{pmatrix} 0 & A^{12} x^2 & 0 \\ 0 & 0 & 0 \\ 0 & A^{32} x^2 & 0 \end{pmatrix} \\ &- \begin{pmatrix} p^1 C^{31} & 0 & 0 \\ 0 & p^2 C^{32} & 0 \\ p^1(I - C^{31}) & p^2(I - C^{32}) & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & A^{13} x^3 \\ 0 & 0 & A^{23} x^3 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{16}$$

where $P^0$ denotes the $(3 \times 3n)$ matrix of price vectors, viz., $P^0 = \begin{pmatrix} p^1 & 0 & 0 \\ 0 & p^2 & 0 \\ 0 & 0 & p^3 \end{pmatrix}$.

Using the commodity-based coefficients $c_{ji}^{rs}$ in (12), the decomposition (15) may be rewritten at the commodity level as

$$\begin{aligned} Y_{ji}^{ss} &= p_i^s c_{ji}^{ss}(\sum_r y_i^{sr} + \sum_{r \neq s} \sum_{j'} a_{ij'}^{sr} x_{j'}^r) - \sum_{r \neq s} p^r (1 - c_{ji}^{sr}) \sum_{j'} a_{ij'}^{rs} x_{j'}^s, \\ Y_{ji}^{rs} &= p_i^s c_{ji}^{rs}(\sum_r y_i^{sr} + \sum_{r \neq s} \sum_{j'} a_{ij'}^{sr} x_{j'}^r) - p_i^r c_{ji}^{sr} \sum_{j'} a_{ij'}^{rs} x_{j'}^s, \quad (r \neq s), \end{aligned} \tag{17}$$

where $Y_{ji}^{rs}$ is the part of the income of country $s$ arising from producing commodity $i$ that is eventually attributable to sector $j$ in country $r$.

# 5 The monetary representation

Koopman et al. (2014) demonstrated a similar measure that evaluates the value added attributable to each country after all the repercussions of trade. Since their results are derived

[1]It can readily be seen $Y^1 = Y^{11} + Y^{21}$, since $c^{11} + c^{21} = 1$ and $c^{22} + c^{12} = 1$ by definition.

from the monetary table, it is important to confirm that our results are consistent with theirs when our formulas are transformed into monetary terms.

The relationship between the physical and monetary inter-regional input coefficients, $a_{ij}^{rs}$ and $\alpha_{ij}^{rs}$, can be established:

$$\alpha_{ij}^{rs} = \frac{p_i^r x_{ij}^{rs}}{p_j^s X_j^s} = \frac{p_i^r}{p_j^s} a_{ij}^{rs}. \tag{18}$$

Similarly, with the physical primary input $L_j^s$, the relationship for the value-added input coefficients, $a_{0j}^s$ and $\alpha_{0j}^s$, results in

$$\alpha_{0j}^{s} = \frac{w^s L_j^s}{p_j^s X_j^s} = \frac{w^s}{p_j^s} a_{0j}^{s}. \tag{19}$$

For simplicity, let us consider a two-country, two-commodity monetary table as shown in Table 1. By using the physical and monetary input coefficients, the output equation for the first line of the table can be written as

$$\begin{aligned} p_1^1 X_1^1 &= p_1^1 a_{11}^{11} X_1^1 + p_1^1 a_{12}^{11} X_2^1 + p_1^1 a_{11}^{12} X_1^2 + p_1^1 a_{12}^{12} X_2^2 + p_1^1 y_1^{11} + p_1^1 y_1^{12} \\ &= p_1^1 \alpha_{11}^{11} X_1^1 + p_2^1 \alpha_{12}^{11} X_2^1 + p_1^2 \alpha_{11}^{12} X_1^2 + p_2^2 \alpha_{12}^{12} X_2^2 + p_1^1 y_1^{11} + p_1^1 y_1^{12} \\ &= \alpha_{11}^{11} \hat{X}_1^1 + \alpha_{12}^{11} \hat{X}_2^1 + \alpha_{11}^{12} \hat{X}_1^2 + \alpha_{12}^{12} \hat{X}_2^2 + \hat{y}_1^{11} + \hat{y}_1^{12} = \hat{X}_1^1, \end{aligned}$$

where $\hat{X}_i^r = p_i^r X_i^r$ and $\hat{y}_i^{rs} = p_i^r y_i^{rs}$ represent the monetary values of $X_i^r$ and $y_i^{rs}$, respectively.

Table 1: Framework for a two-country, two-commodity monetary table.

| | Country 1 | | Country 2 | | Final demand |
|---|---|---|---|---|---|
| Country 1 | $p_1^1 x_{11}^{11}$ | $p_1^1 x_{12}^{11}$ | $p_1^1 x_{11}^{12}$ | $p_1^1 x_{12}^{12}$ | $p_1^1 y_1^{11} + p_1^1 y_1^{12}$ |
| | $p_2^1 x_{21}^{11}$ | $p_2^1 x_{22}^{11}$ | $p_2^1 x_{21}^{12}$ | $p_2^1 x_{22}^{12}$ | $p_2^1 y_2^{11} + p_2^1 y_2^{12}$ |
| Country 2 | $p_1^2 x_{11}^{21}$ | $p_1^2 x_{12}^{21}$ | $p_1^2 x_{11}^{22}$ | $p_1^2 x_{12}^{22}$ | $p_1^2 y_1^{21} + p_1^2 y_1^{22}$ |
| | $p_2^2 x_{21}^{21}$ | $p_2^2 x_{22}^{21}$ | $p_2^2 x_{21}^{22}$ | $p_2^2 x_{22}^{22}$ | $p_2^2 y_2^{21} + p_2^2 y_2^{22}$ |
| Values added | $w^1 a_{01}^1 X_1^1$ | $w^1 a_{02}^1 X_2^1$ | $w^2 a_{01}^2 X_1^2$ | $w^2 a_{02}^2 X_2^2$ | |

Denoting the $4 \times 4$ matrix of inter-regional monetary input coefficients by $\hat{A}$, the system of output equations in the above table can be summarized as

$$\hat{A}\hat{X} + \hat{y} = \hat{X}, \tag{20}$$

where $\hat{X}$ and $\hat{y}$ are the column vectors of monetary outputs and final demands, respectively. To clarify the relationship between monetary and physical expressions, it is necessary to establish the relationship between Leontief inverse matrices in monetary and physical terms. In the two-country and two-commodity setting, the monetary inverse can be transformed as follows:

$$(I - \hat{A})^{-1} = \begin{pmatrix} \beta_{11}^{11} & \beta_{12}^{11} & \beta_{11}^{12} & \beta_{12}^{12} \\ \beta_{21}^{11} & \beta_{22}^{11} & \beta_{21}^{12} & \beta_{22}^{12} \\ \beta_{11}^{21} & \beta_{12}^{21} & \beta_{11}^{22} & \beta_{12}^{22} \\ \beta_{21}^{21} & \beta_{22}^{21} & \beta_{21}^{22} & \beta_{21}^{22} \end{pmatrix} = \begin{pmatrix} 1 - \alpha_{11}^{11} & -\alpha_{12}^{11} & -\alpha_{11}^{12} & -\alpha_{12}^{12} \\ -\alpha_{21}^{11} & 1 - \alpha_{22}^{11} & -\alpha_{21}^{12} & -\alpha_{22}^{12} \\ -\alpha_{11}^{21} & -\alpha_{12}^{21} & 1 - \alpha_{11}^{22} & -\alpha_{12}^{22} \\ -\alpha_{21}^{21} & -\alpha_{22}^{21} & -\alpha_{21}^{22} & 1 - \alpha_{21}^{22} \end{pmatrix}^{-1}$$

$$
\begin{aligned}
&= \begin{pmatrix}
1-\frac{p_1^1}{p_1^1}a_{11}^{11} & -\frac{p_1^1}{p_2^1}a_{12}^{11} & -\frac{p_1^1}{p_1^2}a_{11}^{12} & -\frac{p_1^1}{p_2^2}a_{12}^{12} \\
-\frac{p_2^1}{p_1^1}a_{21}^{11} & 1-\frac{p_2^1}{p_2^1}a_{22}^{11} & -\frac{p_2^1}{p_1^2}a_{21}^{12} & -\frac{p_2^1}{p_2^2}a_{22}^{12} \\
-\frac{p_1^2}{p_1^1}a_{11}^{21} & -\frac{p_1^2}{p_2^1}a_{12}^{21} & 1-\frac{p_1^2}{p_1^2}a_{11}^{22} & -\frac{p_1^2}{p_2^2}a_{12}^{22} \\
-\frac{p_2^2}{p_1^1}a_{21}^{21} & -\frac{p_2^2}{p_2^1}a_{22}^{21} & -\frac{p_2^1}{p_1^2}a_{21}^{22} & 1-\frac{p_2^2}{p_2^2}a_{21}^{22}
\end{pmatrix}^{-1} \\
&= \left( \begin{pmatrix}
p_1^1 & 0 & 0 & 0 \\
0 & p_2^1 & 0 & 0 \\
0 & 0 & p_1^2 & 0 \\
0 & 0 & 0 & p_2^2
\end{pmatrix}
\begin{pmatrix}
1-a_{11}^{11} & -a_{12}^{11} & -a_{11}^{12} & -a_{12}^{12} \\
-a_{21}^{11} & 1-a_{22}^{11} & -a_{21}^{12} & -a_{22}^{12} \\
-a_{11}^{21} & -a_{12}^{21} & 1-a_{11}^{22} & -a_{12}^{22} \\
-a_{21}^{21} & -a_{22}^{21} & -a_{21}^{22} & 1-a_{21}^{22}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{p_1^1} & 0 & 0 & 0 \\
0 & \frac{1}{p_2^1} & 0 & 0 \\
0 & 0 & \frac{1}{p_1^2} & 0 \\
0 & 0 & 0 & \frac{1}{p_2^2}
\end{pmatrix} \right)^{-1} \\
&= \left(P(I-A)P^{-1}\right)^{-1} = P(I-A)^{-1}P^{-1} = P \begin{pmatrix}
b_{11}^{11} & b_{12}^{11} & b_{11}^{12} & b_{12}^{12} \\
b_{21}^{11} & b_{22}^{11} & b_{21}^{12} & b_{22}^{12} \\
b_{11}^{21} & b_{12}^{21} & b_{11}^{22} & b_{12}^{22} \\
b_{21}^{21} & b_{22}^{21} & b_{21}^{22} & b_{21}^{22}
\end{pmatrix} P^{-1}, \qquad (21)
\end{aligned}
$$

where $P$ denotes the $2 \times 2$ diagonal matrix of prices.

Reciprocally, the physical inverse matrix $B$ can also be written in terms of monetary inverse matrix $\hat{B}$:[2]

$$B = (I - A)^{-1} = P^{-1}(I - \hat{A})^{-1}P = P^{-1}\hat{B}P. \qquad (21)'$$

Accordingly, the expressions in (12) can easily be rewritten using monetary coefficients:

$$c_{ji}^{rs} = \frac{w^r a_{0j}^r b_{ji}^{rs}}{p_i^s} = \frac{w^r}{p_i^s}(\frac{p_j^r}{w^r})\alpha_{0j}^r(\frac{p_i^s}{p_j^r})\beta_{ji}^{rs} = \alpha_{0j}^r\beta_{ji}^{rs} \quad \text{and} \quad c_i^{rs} = \sum_j \alpha_{0j}^r\beta_{ji}^{rs}. \qquad (22)$$

Then the GDP decompositions may be calculated by plugging these coefficients into (17).

While Koopman et al. (2014) illustrates the case with single commodity, their approach can easily be extended to the case with multiple commodities. The domestic value-added coefficient $v_j^s$ for sector $j$ corresponds to $\alpha_{0j}^s$ in our notation. Recalling that an element of the monetary Leontief inverse is denoted by $\beta_{ij}^{rs}$, their country shares of values added are calculated for the two commodity case as follows:

$$\begin{pmatrix}
v_1^1 & 0 & 0 & 0 \\
0 & v_2^1 & 0 & 0 \\
0 & 0 & v_1^2 & 0 \\
0 & 0 & 0 & v_2^2
\end{pmatrix}
\begin{pmatrix}
\beta_{11}^{11} & \beta_{12}^{11} & \beta_{11}^{12} & \beta_{12}^{12} \\
\beta_{21}^{11} & \beta_{22}^{11} & \beta_{21}^{12} & \beta_{22}^{12} \\
\beta_{11}^{21} & \beta_{12}^{21} & \beta_{11}^{22} & \beta_{12}^{22} \\
\beta_{21}^{21} & \beta_{22}^{21} & \beta_{21}^{22} & \beta_{22}^{22}
\end{pmatrix}
=
\begin{pmatrix}
\alpha_{01}^1\beta_{11}^{11} & \alpha_{01}^1\beta_{12}^{11} & \alpha_{01}^1\beta_{11}^{12} & \alpha_{01}^1\beta_{12}^{12} \\
\alpha_{02}^1\beta_{21}^{11} & \alpha_{02}^1\beta_{22}^{11} & \alpha_{02}^1\beta_{21}^{12} & \alpha_{02}^1\beta_{22}^{12} \\
\alpha_{01}^2\beta_{11}^{21} & \alpha_{01}^2\beta_{12}^{21} & \alpha_{01}^2\beta_{11}^{22} & \alpha_{01}^2\beta_{12}^{22} \\
\alpha_{02}^2\beta_{21}^{21} & \alpha_{02}^2\beta_{22}^{21} & \alpha_{02}^2\beta_{21}^{22} & \alpha_{02}^2\beta_{22}^{22}
\end{pmatrix} \qquad (23)$$

Let $\hat{p}_j^s$ denote the dual variable for the monetary system. Then by definition, it will be unity, and is calculated as follows:

$$\hat{p}_j^s = \sum_i \sum_r \alpha_{0i}^r \beta_{ij}^{rs} = 1.$$

Hence, the column sums of (23) must be equal to one, and each element represents the share of the value added eventually attributable to the relevant sector and country.

[2]Considering that $\hat{X} = PX$ and $\hat{y} = Py$, the monetary output equation (20) can be written as

$$P(I - A)^{-1}P^{-1}Py = P(I - A)^{-1}y = PX.$$

By pre-multiplying $P^{-1}$, this becomes equivalent to its physical counterpart.

# 6 Exchange rate

Returning to the physical system, it is possible to calculate the effective exchange rate from the balance of payments. If there are $m$ countries, one currency must be regarded as the numéraire, and other currencies are valued relative to it. In the two country case, it is reasonable to regard the currency of country 1 as the numéraire, and let $\mu$ denote the exchange rate for country 2. Then the price equations for each country can be written as

$$\begin{aligned} p^1 A^{11} + \mu p^2 A^{21} + w^1 a_0^1 &= p^1, \\ p^1 A^{12} + \mu p^2 A^{22} + \mu w^2 a_0^2 &= \mu p^2. \end{aligned} \tag{24}$$

By limiting the number of sectors to 2, for simplicity, the trade balance of country 1 can be written as

$$\begin{aligned} & p_1^1 a_{11}^{12} x_1^2 + p_2^1 a_{21}^{12} x_1^2 + p_1^1 a_{12}^{12} x_2^2 + p_2^1 a_{22}^{12} x_2^2 + p_1^1 y_1^{12} + p_2^1 y_2^{12} \\ - \; & \mu (p_1^2 a_{11}^{21} x_1^1 + p_2^2 a_{21}^{21} x_1^1 + p_1^2 a_{12}^{21} x_2^1 + p_2^2 a_{22}^{21} x_2^1 + p_1^2 y_1^{21} + p_2^2 y_2^{21}) = 0. \end{aligned}$$

If there is no income transfer or capital flow between the two countries, the exchange rate $\mu$ is determined solely from the above. However, this is unlikely so a net capital flow $F$ into country 1 is introduced, and the equation is modified to include $F$:[3]

$$\begin{aligned} & p_1^1 a_{11}^{12} x_1^2 + p_2^1 a_{21}^{12} x_1^2 + p_1^1 a_{12}^{12} x_2^2 + p_2^1 a_{22}^{12} x_2^2 + p_1^1 y_1^{12} + p_2^1 y_2^{12} + F \\ - \; & \mu (p_1^2 a_{11}^{21} x_1^1 + p_2^2 a_{21}^{21} x_1^1 + p_1^2 a_{12}^{21} x_2^1 + p_2^2 a_{22}^{21} x_2^1 + p_1^2 y_1^{21} + p_2^2 y_2^{21}) = 0. \end{aligned} \tag{25}$$

In a world with only two countries, the balance of payments for country 2, where the capital flow is given by $-\mu F$, brings no additional information. Then the exchange rate can directly be calculated from (25):

$$\mu = \frac{p_1^1 a_{11}^{12} x_1^2 + p_2^1 a_{21}^{12} x_1^2 + p_1^1 a_{12}^{12} x_2^2 + p_2^1 a_{22}^{12} x_2^2 + p_1^1 y_1^{12} + p_2^1 y_2^{12} + F}{p_1^2 a_{11}^{21} x_1^1 + p_2^2 a_{21}^{21} x_1^1 + p_1^2 a_{12}^{21} x_2^1 + p_2^2 a_{22}^{21} x_2^1 + p_1^2 y_1^{21} + p_2^2 y_2^{21}}. \tag{26}$$

In the present framework, where the final demands in physical units are given exogenously, the physical outputs can be determined independent of the price system. Thus the solution to the problem (7) can readily be found:

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} I - A^{11} & -A^{12} \\ -A^{21} & I - A^{22} \end{pmatrix}^{-1} \begin{pmatrix} y^{11} + y^{12} \\ y^{21} + y^{22} \end{pmatrix}. \tag{27}$$

However, monetary variables $w^r$ and $p_i^r$ are to be determined through an iterative process. When wages $(w^1, w^2)$ are appropriately given, the corresponding price vectors $(p^1, p^2)$ are calculated by (10). Then, given capital flow $F$, the initial exchange rate $\tilde{\mu}^{(0)}$ is determined by (26). While the wages must be evaluated in the local currency, our initial setup is denominated in the common currency. Hence, the wage in country 2 must be revised to reflect the provisional exchange rate $\mu = \tilde{\mu}^{(0)}$, in step $k = 1$. Thus

$$(p^1 \quad p^2) = (w^1 a_0^1 \quad \mu w^2 a_0^2) \begin{pmatrix} I - A^{11} & -A^{12} \\ -A^{21} & I - A^{22} \end{pmatrix}^{-1} = \begin{pmatrix} w^1 a_0^1 B^{11} & w^1 a_0^1 B^{12} \\ \mu w^2 a_0^2 B^{21} & \mu w^2 a_0^2 B^{22} \end{pmatrix}.$$

[3]To be exact, the balance of payments is obtained as the sum of trade balance, income transfer, and capital flows. Here the sum of the latter two is simply called "capital flow".

When these revised price vectors are plugged into (26), the incremental exchange rate $\tilde{\mu}^{(k)}$ is obtained. Convergence is reached when $|\tilde{\mu}^{(k)} - 1| < \epsilon$ is satisfied with sufficiently small $\epsilon > 0$. Otherwise, the above process must be repeated with the exchange rate $\mu = \prod_{i=0}^{k} \tilde{\mu}^{(i)}$ in step $k+1$. If the process converged at step $\ell$, the exchange rate and corresponding country 2's wage in the local currency are, respectively, given by

$$\mu = \prod_{k=1}^{\ell} \tilde{\mu}^{(i)} \quad \text{and} \quad \hat{w}^2 = \mu w^2.$$

When there are $m > 2$ countries, $m-1$ independent exchange rates are determined. The balance of payments for country $r$ can then be written as

$$\mu^r \sum_i p_i^r \sum_{s \neq r} (\sum_j a_{ij}^{rs} x_j^s + y_i^{rs}) = \sum_{s \neq r} \mu^s \sum_i p_i^s (\sum_j a_{ij}^{sr} x_j^r + y_i^{sr}) \tag{28}$$

By letting $\mu^1 = 1$, the rest of exchange rates $\mu^r (r = 2, \ldots, m)$ can be found using a similar iterative process to the one described above. In any case, it must be emphasized that the exchange rates depend crucially on how the wage levels in individual countries and capital flows among them are specified.

# 7 GHG emissions

Consider a world of two countries where GHG emissions are not priced. The output system can be written in exactly the same way as the constraints in problem (7):

$$\begin{aligned} (I - A^{11})x^1 - A^{12}x^2 &= y^{11} + y^{12}, \\ -A^{21}x^1 + (I - A^{22})x^2 &= y^{21} + y^{22}. \end{aligned}$$

Let $a_g^r$ be the unit emission vector from production activities, and $e_g^r$ be the same from consumption of final products in country $r$.[4] Then the emission in each country is calculated as follows:[5]

$$g^1 = a_g^1 x^1 + e_g^1 (y^{11} + y^{21}) \quad \text{and} \quad g^2 = a_g^2 x^2 + e_g^2 (y^{12} + y^{22}).$$

Since the Leontief inverse represents the infinite repercussions of inter-sector and international transactions, it is straightforward to assess the impact of each final demand segment on the GHG emissions of each country:

$$\begin{aligned} \begin{pmatrix} g^1 \\ g^2 \end{pmatrix} &= \begin{pmatrix} a_g^1 & 0 \\ 0 & a_g^2 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} e_g^1 & e_g^1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y^{11} \\ y^{21} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ e_g^2 & e_g^2 \end{pmatrix} \begin{pmatrix} y^{12} \\ y^{22} \end{pmatrix} \\ &= \begin{pmatrix} a_g^1 B^{11} + e_g^1 & a_g^1 B^{12} + e_g^1 \\ a_g^2 B^{21} & a_g^2 B^{22} \end{pmatrix} \begin{pmatrix} y^{11} \\ y^{21} \end{pmatrix} + \begin{pmatrix} a_g^1 B^{11} & a_g^1 B^{12} \\ a_g^2 B^{21} + e_g^2 & a_g^2 B^{22} + e_g^2 \end{pmatrix} \begin{pmatrix} y^{12} \\ y^{22} \end{pmatrix}. \end{aligned} \tag{29}$$

[4] The seminal article by Leontief (1970) considers pollutants from only production sectors. However, GHG emissions from the final demand sectors cannot be ignored.

[5] When only one gas is being considered, $a_g^r$ becomes a row vector of size $n$, but the method can easily be extended to cover $k$ kinds of gas; in that case, $a_g^r$ becomes a $k \times n$ matrix. Moreover, the same formulation can also be applied to water resources. In that case, $a_g^r$ and $e_g^r$ are interpreted as the unit water demand associated with the production process, and with the final demand consumption, respectively.

Each country is responsible for the emissions accrued from its final demand. For example, country 1's emission $g^1$ can be decomposed into two parts, $g^{11}$ and $g^{12}$, for which countries 1 and 2, respectively, are responsible.

$$g^{11} = (a_g^1 B^{11} + e_g^1)y^{11} + (a_g^1 B^{12} + e_g^1)y^{21} \quad \text{and} \quad g^{12} = a_g^1 B^{11} y^{12} + a_g^1 B^{12} y^{22}.$$

Likewise, country 2's emission $g^2$ can also be decomposed:

$$g^{21} = a_g^2 B^{21} y^{11} + a_g^2 B^{22} y^{21} \quad \text{and} \quad g^{22} = (a_g^2 B^{21} + e_g^2)y^{12} + (a_g^2 B^{22} + e_g^2)y^{22}.$$

As with $c_i^{rs}$ in (12), it is possible to define the fraction $f^{rs}$ of gas emissions in country $r$, for which country $s$ is responsible, in a multi-country setting as follows:[6]

$$f^{rr} = \frac{\sum_\ell (a_g^r B^{r\ell} + e_g^r) y^{\ell r}}{a_g^r x^r + e_g^r \sum_\ell y^{\ell r}} \quad \text{and} \quad f^{rs} = \frac{a_g^r \sum_\ell B^{r\ell} y^{\ell s}}{a_g^r x^r + e_g^r \sum_\ell y^{\ell r}} \quad (r \neq s) \tag{30}$$

with $\sum_s f^{rs} = 1$ being satisfied by definition. Alternately, the above expressions can be detailed to the commodity level:

$$f^{rr} = \frac{\sum_i (a_{gi}^r \sum_\ell \sum_j b_{ij}^{r\ell} y_j^{\ell r} + e_{gi}^r \sum_\ell y_i^{\ell r})}{\sum_i (a_{gi}^r x_i^r + e_{gi}^r \sum_\ell y_i^{\ell r})} \quad \text{and} \quad f^{rs} = \frac{\sum_i a_{gi}^r \sum_\ell \sum_j b_{ij}^{r\ell} y_j^{\ell s}}{\sum_i (a_{gi}^r x_i^r + e_{gi}^r \sum_\ell y_i^{\ell r})} \quad (r \neq s) \tag{31}$$

In matrix form, equation (29) can easily be extended to the multi-country case by defining the following matrices:

$$A_g = \begin{pmatrix} a_g^1 & 0 & \cdots & 0 \\ 0 & a_g^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_g^m \end{pmatrix}, B = \begin{pmatrix} B^{11} & \cdots & B^{1m} \\ \vdots & \ddots & \vdots \\ B^{m1} & \cdots & B^{mm} \end{pmatrix}, E_g = \begin{pmatrix} e_g^1 & e_g^1 & \cdots & e_g^1 \\ e_g^2 & e_g^2 & \cdots & e_g^2 \\ \vdots & \vdots & \ddots & \vdots \\ e_g^m & e_g^m & \cdots & e_g^m \end{pmatrix},$$

where $A_g$, $B$ and $E_g$ are matrices of sizes $m \times mn$, $mn \times mn$, and $m \times mn$, respectively. Further, define

$$X = \begin{pmatrix} x^1 \\ \vdots \\ x^m \end{pmatrix}, Y = \begin{pmatrix} y^{11} & \cdots & y^{1m} \\ \vdots & \ddots & \vdots \\ y^{m1} & \cdots & y^{mm} \end{pmatrix}, g = \begin{pmatrix} g^1 \\ \vdots \\ g^m \end{pmatrix}, \mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

which are a column vector of size $mn$, a matrix of size $mn \times m$, a column vector of size $m$, and the all-one vector of size $m$, respectively. Then the decomposition of GHG emissions from the production process can be written as

$$A_g X = A_g B Y \mathbf{1}.$$

With the operator $Diag(\bullet)$ to extract the diagonal elements of square matrices, the emissions from final demand consumption can be written as $Diag(E_g Y)\mathbf{1}$. Thus the emission vector $G$ can be written, in matrix form, as

$$g = (A_g B Y + Diag(E_g Y))\mathbf{1}. \tag{32}$$

[6]In the case of price decomposition, $c_i^{rs}$ represents the share of product $i$ in country $s$ that comes from country $r$. When comparing the sums $\sum_r c_i^{rs} = 1$ and $\sum_s f^{rs} = 1$, the superscripts appear to be reversed as they represent transfers in opposite directions: TiVA represents backward linkage while emissions respond to forward linkage.

The decomposition of GHG emissions over the countries can then be obtained using (30):

$$G = \begin{pmatrix} g^{11} & g^{12} & \cdots & g^{1m} \\ g^{21} & g^{22} & \cdots & g^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ g^{m1} & g^{m2} & \cdots & g^{mm} \end{pmatrix} = \begin{pmatrix} g^{1} & 0 & \cdots & 0 \\ 0 & g^{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g^{m} \end{pmatrix} \begin{pmatrix} f^{11} & f^{12} & \cdots & f^{1m} \\ f^{21} & f^{22} & \cdots & f^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ f^{m1} & f^{m2} & \cdots & f^{mm} \end{pmatrix} = diag(g)F. \tag{33}$$

# 8 The problem with GHG abatement

The model in the previous section is open-ended in the sense that it simply calculates the GHG emissions and clarifies the responsibility of each country without considering environmental restrictions. In contrast, when such restrictions and pollution abatement activity are introduced, it is possible to assess the fair penalty for the GHG emissions discharged into the environment. In the case of an isolated country with two industrial sectors and an abatement sector, the environmental restriction is normally given in the form

$$a_{g1}x_1 + a_{g2}x_2 + a_{gg}x_g + e_{g1}y_1 + e_{g2}y_2 - x_g \leq g, \tag{34}$$

where $g$ is the amount of GHG permitted to be discharged into the environment, $x_g$ is the amount of GHG eliminated, and $a_{gg}$ is the GHG emission from the abatement activity. Likewise the output requirement for industrial sectors can be written using the input requirement for a unit reduction of GHG, $a_{ig}$, as follows:

$$x_i - a_{i1}x_1 - a_{i2}x_2 - a_{ig}x_g \geq y_i \quad (i = 1, 2).$$

Considering the direction of inequalities, a linear programming problem similar to (1) can be formulated with labor input $a_{0g}$ in the abatement sector:

$$\begin{array}{ll} \min_{x_1,x_2,x_g} & w(a_{01}x_1 + a_{02}x_2 + a_{0g}x_g), \\ \text{s.t.} & (1-a_{11})x_1 - a_{12}x_2 - a_{1g}x_g \geq y_1, \\ & -a_{21}x_1 + (1-a_{22})x_2 - a_{2g}x_g \geq y_2, \\ & -a_{g1}x_1 - a_{g2}x_2 + (1-a_{gg})x_g \geq e_{g1}y_1 + e_{g2}y_2 - g, \\ & x_1, x_2, x_g \geq 0. \end{array} \tag{35}$$

The solution to this problem can be readily obtained:[7]

$$\begin{pmatrix} x_1 \\ x_2 \\ x_g \end{pmatrix} = \begin{pmatrix} 1-a_{11} & -a_{12} & -a_{1g} \\ -a_{21} & 1-a_{22} & -a_{2g} \\ -a_{g1} & -a_{g2} & 1-a_{gg} \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \\ e_{g1}y_1 + e_{g2}y_2 - g \end{pmatrix}. \tag{36}$$

[7] According to the weak-solvability condition, the solution to the Leontief model, $x = (I-A)^{-1}y$, is guaranteed non-negative when the Leontief matrix $(I-A)$ is positive definite and $y$ is non-negative (see, e.g., Nikaido, 1968). In this case, however, such conditions do not necessarily apply since $e_{g1}y_1 + e_{g2}y_2 - g$ could be negative in an unrealistic case where the environmental restriction is very loose and there is no need for abatement.

When $\lambda$ denotes the Lagrangian multiplier assigned to (34), it is interpreted as the unit price of GHG emission. All the price variables, including $\lambda$, are obtained from the dual system:

$$(p_1 \quad p_2 \quad \lambda) = w(a_{01} \quad a_{02} \quad a_{0g}) \begin{pmatrix} 1-a_{11} & -a_{12} & -a_{1g} \\ -a_{21} & 1-a_{22} & -a_{2g} \\ -a_{g1} & -a_{g2} & 1-a_{gg} \end{pmatrix}^{-1}. \tag{37}$$

To extend this approach to the problem of a world comprising two countries, the output equations are formulated for individual countries as follows:

$$\begin{pmatrix} x_1^1 \\ x_2^1 \\ x_g^1 \end{pmatrix} - \begin{pmatrix} a_{11}^{11} & a_{12}^{11} & a_{1g}^{11} \\ a_{21}^{11} & a_{22}^{11} & a_{2g}^{11} \\ a_{g1}^{1} & a_{g2}^{1} & a_{gg}^{1} \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_g^1 \end{pmatrix} - \begin{pmatrix} a_{11}^{12} & a_{12}^{12} & a_{1g}^{12} \\ a_{21}^{12} & a_{22}^{12} & a_{2g}^{12} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_g^2 \end{pmatrix} = \begin{pmatrix} y_1^{11} + y_1^{12} \\ y_2^{11} + y_2^{12} \\ e_{g1}^1(y_1^{11} + y_1^{21}) + e_{g2}^1(y_2^{11} + y_2^{21}) - g^1 \end{pmatrix},$$

$$\begin{pmatrix} x_1^2 \\ x_2^2 \\ x_g^2 \end{pmatrix} - \begin{pmatrix} a_{11}^{21} & a_{12}^{21} & a_{1g}^{21} \\ a_{21}^{21} & a_{22}^{21} & a_{2g}^{21} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_g^1 \end{pmatrix} - \begin{pmatrix} a_{11}^{22} & a_{12}^{22} & a_{1g}^{22} \\ a_{21}^{22} & a_{22}^{22} & a_{2g}^{22} \\ a_{g1}^{2} & a_{g2}^{2} & a_{gg}^{2} \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_g^2 \end{pmatrix} = \begin{pmatrix} y_1^{21} + y_1^{22} \\ y_2^{21} + y_2^{22} \\ e_{g1}^2(y_1^{12} + y_1^{22}) + e_{g2}^2(y_2^{12} + y_2^{22}) - g^2 \end{pmatrix}$$

Here transportation of GHG across countries is precluded: i.e., production activity in one country does not discharge GHG in the other country.[8]

For the sake of convenience, let $\tilde{A}^{11}, \tilde{A}^{12}, \tilde{A}^{21}$ and $\tilde{A}^{22}$, respectively, denote the matrices of input coefficients in the order that they appeared in the above two equations. Also let $u^1$ and $u^2$ denote the column vectors on the right hand sides of the above equations. Moreover, define the augmented column vector of outputs and row vector of labor inputs as follows:

$$\tilde{x}^r = (x_1^r \quad x_2^r \quad x_g^r)' \quad \text{and} \quad \tilde{a}_0^r = (a_{01}^r \quad a_{02}^r \quad a_{0g}^r).$$

Then the world problem with GHG abatement activity can be formulated in matrix form:

$$\begin{aligned} \min_{\tilde{x}^1, \tilde{x}^2} \quad & w^1\tilde{a}_0^1\tilde{x}^1 + w^2\tilde{a}_0^2\tilde{x}^2 \\ \text{s.t.} \quad & (I - \tilde{A}^{11})\tilde{x}^1 - \tilde{A}^{12}\tilde{x}^2 \geq u^1 && (38) \\ & -\tilde{A}^{21}\tilde{x}^1 + (I - \tilde{A}^{22})\tilde{x}^2 \geq u^2 && (39) \\ & \tilde{x}^1, \tilde{x}^2 \geq 0. \end{aligned}$$

By denoting the row vectors of Lagrange multipliers attached to (38) and (39) as $q^1$ and $q^2$, respectively, the Lagrangian function for the problem can be written as

$$L = w^1\tilde{a}_0^1\tilde{x}^1 + w^2\tilde{a}_0^2\tilde{x}^2 + q^1(u^1 - (I - \tilde{A}^{11})\tilde{x}^1 + \tilde{A}^{12}\tilde{x}^2) + q^2(u^2 + \tilde{A}^{21}\tilde{x}^1 - (I - \tilde{A}^{22})\tilde{x}^2). \tag{40}$$

With non-negativity constraints, the first-order conditions would become

$$\begin{aligned} \frac{\partial L}{\partial \tilde{x}^1} &= w^1\tilde{a}_0^1 - q^1(I - \tilde{A}^{11}) + q^2\tilde{A}^{21} \geq 0, \\ \frac{\partial L}{\partial \tilde{x}^2} &= w^2\tilde{a}_0^2 + q^1\tilde{A}^{12} - q^2(I - \tilde{A}^{22}) \geq 0. && (41) \end{aligned}$$

[8] It must be noted that the combination of traded final demands for GHG is different from that of other commodities.

Thus the multipliers are determined as follows:

$$(q^1 \quad q^2) = (p_1^1 \;\; p_2^1 \;\; \lambda^1 \;\; p_1^2 \;\; p_2^2 \;\; \lambda^2) \leq (w^1\tilde{a}_0^1 \quad w^2\tilde{a}_0^2)\begin{pmatrix} I-\tilde{A}^{11} & -\tilde{A}^{12} \\ -\tilde{A}^{21} & I-\tilde{A}^{22} \end{pmatrix}^{-1}. \tag{42}$$

When the vector $(\tilde{x}^1, \tilde{x}^2)$ is positive, (42) holds with equality. However, this is not necessarily true with very loose environmental restrictions since GHG emission may become a "free good" ($\lambda^r = 0$) in that case.

Suppose all the constraints are binding, and $\tilde{B}^{rs}$ denotes an element of the Leontief inverse in (42). Then the responsibilities for GHG emissions are distributed over the countries in the same way as in the case without abatement activity.

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} (\tilde{a}_g^1\tilde{B}^{11} + \tilde{e}_g^1)\tilde{y}^{11} + (\tilde{a}_g^1\tilde{B}^{12} + \tilde{e}_g^1)\tilde{y}^{21} & \tilde{a}_g^1\tilde{B}^{11}\tilde{y}^{12} + \tilde{a}_g^1\tilde{B}^{12}\tilde{y}^{22} \\ \tilde{a}_g^2\tilde{B}^{21}\tilde{y}^{11} + \tilde{a}_g^2\tilde{B}^{22}\tilde{y}^{21} & (\tilde{a}_g^2\tilde{B}^{21} + \tilde{e}_g^2)\tilde{y}^{12} + (\tilde{a}_g^2\tilde{B}^{22} + \tilde{e}_g^2)\tilde{y}^{22} \end{pmatrix}, \tag{43}$$

where vectors $\tilde{a}_g^r, \tilde{e}_g^r$, and $\tilde{y}^{rs}$ are also augmented to include the abatement.

$$\tilde{a}_g^r = (a_{g1}^r \;\; a_{g2}^r \;\; a_{gg}^r), \quad \tilde{e}_g^r = (e_{g1}^r \;\; e_{g2}^r \;\; 0), \quad \text{and} \quad \tilde{y}^{rs} = (y_1^{rs} \;\; y_2^{rs} \;\; 0)'.$$

By the same token, the values added can also be decomposed:

$$\begin{pmatrix} Y^{11} & Y^{12} \\ Y^{21} & Y^{22} \end{pmatrix} =$$
$$\begin{pmatrix} q^1\tilde{C}^{11}(\tilde{y}^{11} + \tilde{y}^{12} + \tilde{A}^{12}\tilde{x}^2) - q^2\tilde{C}^{22}\tilde{A}^{21}\tilde{x}^1 & q^1\tilde{C}^{12}(\tilde{y}^{21} + \tilde{y}^{22} + \tilde{A}^{21}\tilde{x}^1) - q^1\tilde{C}^{21}\tilde{A}^{12}\tilde{x}^2 \\ q^2\tilde{C}^{21}(\tilde{y}^{11} + \tilde{y}^{12} + \tilde{A}^{12}\tilde{x}^2) - q^2\tilde{C}^{12}\tilde{A}^{21}\tilde{x}^1 & q^2\tilde{C}^{22}(\tilde{y}^{21} + \tilde{y}^{22} + \tilde{A}^{21}\tilde{x}^1) - q^1\tilde{C}^{11}\tilde{A}^{12}\tilde{x}^2 \end{pmatrix}, \tag{44}$$

where the diagonal matrix $\tilde{C}^{rs}$ is augmented to include the decomposition of GHG abatement cost $\lambda^s$.

In this article, we demonstrated that the linear programming equivalent of a physical IO system can decompose both value added and GHG emissions, down to the level of ultimate beneficiaries or causes, in a consistent manner. The GHG emissions are likely to be proportional to the physical amounts produced or consumed rather than their monetary values. For example, fuel efficiency would better be evaluated by the liters rather than dollars of gasoline burned, and thus, the use of a physical system seems more appropriate. When GHG abatement activity is introduced, the price for emission rights can be endogenized. Then the question is how to determine a fair allocation of emission permits (see Uzawa, 2003). The existence of tradable emission permits then introduces a new form of income transfer among countries. Besides when the domestic labor market and international financial market are properly combined, the physical framework can also endogenize the exchange rates and, by annexing several markets outside the IO system, the system becomes closer to the spatial computable general equilibrium (SCGE) model (see, e.g., Ando and Meng, 2014).

Although the physical IO system has several desirable properties, the problem is that (international) physical tables are not available. Thus one important task is to compile a physical table from existing monetary tables, and derive some meaningful analytical results. However, such a task is beyond the scope of this article, and has been left for the future research.

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