3. Quadratic Programming for Constrained Matrix Balancing

Kazuhiko Oyamada

1. Introduction

In this section, we will discuss an optimization-based flexible approach to reconcile row and column totals of an input-output (IO) table, which entries are filled with data obtained from various sources and/or inconsistent with each other. To demonstrate calculations, the 2015 Input-Output Tables for Japan is utilized for example (Ministry of Internal Affairs and Communications, 2019). Table 1 depicts an image of the IO table, which contains the variables we will use in this study. The original table is aggregated into three commodities/activities, three value-added items, and three final demand items. The three production sectors are primary industry, manufacturing, and services, which are denoted by subscript i or j (S01, S02, S03) in the following discussions. The value-added items are compensation for employees, operating surplus, and indirect taxes/subsidies on production, denoted by subscript v (V01, V02, V03). The three final demand items are consumption plus fixed capital formation, exports, and tax inclusive imports (deductions), denoted by subscript f (F01, F02, F03). The actual IO table is shown in Table 2. Notice that the table includes negative values. The method we will introduce here is capable of handling negative values without any problems. Sometimes, sign reversals like the one from negative to positive or positive to negative can also occur in the estimation process. Of course, such sign reversals can be prevented by placing upper or lower bounds on endogenous variables. On the other hand, zero values should be fixed to zero and excluded from the set of endogenous variables.

To artificially recreate the state before the balancing process, all of the value entries on Table 2 are rounded to the nearest ten thousand as shown in Table 3. These values on Table 3 represent raw data just collected from various sources, and provide initial values of variables that will be estimated to balance the table. Table 4 shows the percentage deviations of the rounded entries on Table 3 from the original values on Table 2.

2. Setting Up Optimization Problems

Let us formulate an objective function to be minimized in a mathematical problem that penalize the divergence between estimated values of data entries, which will make the table balanced, and the present entries on Table 3, with which the table is unbalanced, as follows:

$$(1)\quad \min_{X_{ij}^*,Y_{vj}^*,Z_{if}^*} \Omega = \sum_i \sum_j \left(\frac{X_{ij}^*-X_{ij}}{X_{ij}}\right)^2 + \sum_v \sum_j \left(\frac{Y_{vj}^*-Y_{vj}}{Y_{vj}}\right)^2 + \sum_i \sum_f \left(\frac{Z_{if}^*-Z_{if}}{Z_{if}}\right)^2$$

where

Ω is the objective variable to be minimized,

X_{ij} is initial values of intermediate demand appeared on Table 3,

X_{ij}^* is estimated values of intermediate demand,

Y_{vj} is initial values of value-added items appeared on Table 3,

Y_{vj}^* is estimated values of value-added items,

Z_{if} is initial values of final demand items appeared on Table 3, and

Z_{if}^* is estimated values of final demand items.

Note that the objective function (1) equally penalize the deviations regardless of their magnitude, *i.e.*, the divergence between 1 and 0.5 has the same importance as the one with 1,000,000 and 500,000.

Then, an alternative version of the objective function can be considered:

$$(2)\quad \min_{X_{ij}^*,Y_{vj}^*,Z_{if}^*} \Omega = \sum_i \sum_j |X_{ij}| \left(\frac{X_{ij}^*-X_{ij}}{X_{ij}}\right)^2 + \sum_v \sum_j |Y_{vj}| \left(\frac{Y_{vj}^*-Y_{vj}}{Y_{vj}}\right)^2 + \sum_i \sum_f |Z_{if}| \left(\frac{Z_{if}^*-Z_{if}}{Z_{if}}\right)^2.$$

With this type of objective function, in which the sizes of initial values are chosen as the weights, the deviations for larger entries are preferentially minimized. In this case, the importance of the divergence between 1 and 0.5 corresponds to the one with 500,500 and 500,000. In this study, we will see the differences in the estimation results switching these two types of objective function.[1] In the following, we call these objective functions Type (1) and Type (2), respectively.

To balance an IO table, some sorts of constraints are needed. If we follow the same procedure as in the case of RAS method, one can introduce the following two constraints with information on the so-called "control totals" (CTs) handy:

$$(3)\quad \sum_i X_{ij}^* + \sum_v Y_{vj}^* = \overline{CT}_j,$$

and

$$(4)\quad \sum_j X_{ij}^* + \sum_f Z_{if}^* = \overline{CT}_i.$$

In the experiments, we use the original values of gross output in Table 2 as $\overline{CT}_i$ and $\overline{CT}_j$.

In general, the constraints (3) and (4) are not necessarily necessary to balance an IO table. Even in a case it is difficult to prepare CTs, a table can still be balanced applying the following constraint:

(5) $\sum_i X^*_{ij} + \sum_v Y^*_{vj} = \sum_i X^*_{ji} + \sum_f Z^*_{jf}$.

Equation (5) is the minimum required constraint to form a matrix balancing problem.

One of the greatest advantages of the optimization-based quadratic programming is its flexibility. While Equation (5) is the minimum requirement as noted above, one may add any kind of constraints depending on the availability of data. If the information on gross domestic product (GDP) at market prices is available, one can consider to introduce a constraint as follows:

(6) $\sum_v \sum_j Y^*_{vj} = \overline{GDP}$.

For another example, one can add a constraint that binds the total value of some final demand items to a certain amount:

(7) $\sum_i Z^*_{i"F02"} = \overline{EXP}$.

Equation (7) is an example of fixing the value of total exports to a given level. The values of $\overline{GDP}$ and $\overline{EXP}$ are also calculated from the original values on Table 2 for experiments, similar to the case of CTs.

Eventually, four combinations of constraints with two types of objective function form eight patterns of optimization problems. Those are summarized in Table 5. The main differences between four combinations of constraints are whether (6) and (7) are added to the basic constraint(s) or not. The basic set of constraint(s) is either of the set of (3) and (4) or just (5). In the experiments that follow, processing is implemented by the General Algebraic Modeling System (GAMS), with which the optimization problems are formulated in the NLP format and solved by CONOPT4.[2] Thus, the computational program is just straight-forward coding Equations (1), (3), (4), (6), and (7), in the case of Problem 1D for example, which is solved by declaring "SOLVE <Model Name> MINIMIZING Ω USING NLP." It is not necessary to derive first order conditions (FOCs). GAMS handles an optimization problem as optimization problem when NLP formulation is applied, and directly solves it.

3. Results of Data Processing in NLP Format

Tables 6 through 13 correspond to the results of computation with eight patterns of Cases 1A through 2D. As noted, these are the results with NLP formulations solved by CONOPT4. In every set of tables, the one on the top shows the estimated values, which satisfy the designated constraints, the one in the middle captures the percentage deviations from the original values on Table 2, and the one at the bottom indicates how much the estimated values come closer to the original values by processing.

For instance, the positive/negative values in the table at the bottom implies that the deviations are getting smaller/greater compared to those of initial values appeared on Table 4.

At a first glance, one may notice that there might be a tendency to try to limit the number of entries, which are changed from their initial values, when Type (1) objective is applied (Tables 6 through 9). If CTs are not used, the number of adjusted entries are limited to two and five respectively out of 27 in Cases 1B and 1C (Tables 7 and 8). In fact, this tendency does not change even if other solvers such as CONOPT and MINOS are used for computation. On the other hand, the patterns of estimations (which entries are changed) are completely different among solvers (it can be confirmed later with Tables 16 and 17). This fact strongly suggests that the obtained solution might not be unique with Type (1) objective.

If we compare the results from Cases 1B through 1D (Tables 7 through 9), it can be said that the availability of additional information may not contribute to improve the accuracy of estimation, because more entries move away from their original (target) values on Table 2. Comparing Case 1A with 1B, it still is ambiguous whether additional information helps to improve the estimation accuracy, as the number of entries moving away from the original values also increases when those approaching the target increase.

Let us move to the cases with Type (2) objective (Tables 10 through 13). In these cases, the non-uniqueness of obtained solutions seems to have been resolved. All of the solvers, CONOPT4, CONOPT, and MINOS, provide identical solutions in each case. While the diagonal entries of intermediate transactions (X_{ij}^{*}) still keep sticking to their initial values (X_{ij}) when CTs are not available (Cases 2B and 2C, Tables 11 and 12), almost all of the entries are evenly adjusted through the reconciliation process.

The contribution of additional information to estimation accuracy is yet ambiguous.
While the introduction of CTs seems to be contributing in Cases 2A and 2B (Tables 10 and 11), introducing the constraints related to GDP and total exports does not (Cases 2B and 2C, Tables 11 and 12) or worsens the result (Cases 2A and 2D, Tables 10 and 13).

4. Results of Data Processing in Various Formats

Instead of handling the minimization problem in the NLP format, there are several other approaches. One is to formulate the balancing program as a system of simultaneous equations, which consists of a full set of FOCs for an optimum. Taking Case 1D as an example, let us demonstrate this procedure.

The optimization problem 1D is expressed as follows:

$$\text{(1)} \quad \min_{X_{ij}^*, Y_{vj}^*, Z_{if}^*} \Omega = \sum_i \sum_j \left(\frac{X_{ij}^* - X_{ij}}{X_{ij}}\right)^2 + \sum_v \sum_j \left(\frac{Y_{vj}^* - Y_{vj}}{Y_{vj}}\right)^2 + \sum_i \sum_f \left(\frac{Z_{if}^* - Z_{if}}{Z_{if}}\right)^2$$

$$\text{(3)} \quad \text{s.t.} \quad \sum_i X_{ij}^* + \sum_v Y_{vj}^* = \overline{CT}_j, \qquad \perp \alpha_j$$

$$\text{(4)} \quad \sum_j X_{ij}^* + \sum_f Z_{if}^* = \overline{CT}_i. \qquad \perp \beta_i$$

$$\text{(6)} \quad \sum_v \sum_j Y_{vj}^* = \overline{GDP}, \qquad \perp \gamma$$

$$\text{(7)} \quad \sum_i Z_{i\text{"}F02\text{"}}^* = \overline{EXP}, \qquad \perp \delta$$

where α_j, β_i, γ, and δ are Lagrange multipliers. The perpendicular symbol "⊥" shows counterpart relationships between constraints and endogenous variables. Forming a Lagrangian and partially differentiating it with respect to X_{ij}^*, Y_{vj}^*, and Z_{if}^*, respectively, we obtain the following three FOCs:

$$\text{(8)} \quad \frac{X_{ij}^*}{X_{ij}} + \frac{X_{ij}}{2}\alpha_j + \frac{X_{ij}}{2}\beta_i = 1, \qquad \perp X_{ij}^*$$

$$\text{(9)} \quad \frac{Y_{vj}^*}{Y_{vj}} + \frac{Y_{vj}}{2}\alpha_j + \frac{Y_{vj}}{2}\gamma = 1, \qquad \perp Y_{vj}^*$$

and

$$\text{(10)} \quad \frac{Z_{if}^*}{Z_{if}} + \frac{Z_{if}}{2}\beta_i + \frac{Z_{i\text{"}F02\text{"}}}{2}\delta = 1. \qquad \perp Z_{if}^*$$

Note that the third item in the left-hand side of Equation (10) enters only in the case when $f = \text{"}F02\text{"}$.

Coding seven equations, (3), (4), and (6) through (10), all is set. It can be solved by GAMS declaring "SOLVE <Model Name> USING CNS," or by another application that can handle system of simultaneous equations. Notice that there is no non-linear constraints included in this system. It implies that this system of equations can be expressed in a matrix form as an alternative approach:

$$\text{(11)} \quad \mathbf{\Phi\Xi = \Theta}$$

where

- $\mathbf{\Phi}$ is the 35×35 square matrix that contains coefficients such as $\frac{1}{X_{ij}}$ and $\frac{X_{ij}}{2}$,
- $\mathbf{\Xi}$ is the column vector that contains 35 endogenous variables, X_{ij}^*, Y_{vj}^*, Z_{if}^*, α_j, β_i, γ, and δ, and
- $\mathbf{\Theta}$ is the column vector that contains eight exogenous variables, $\overline{CT}_j$, $\overline{CT}_i$, $\overline{GDP}$, and $\overline{EXP}$, as well as 27 "1"s that enter the right-hand side of Equations (8) through (10).

Then, this system of equations can be solved multiplying the inverse of matrix $\mathbf{\Phi}$ from the left in both hands of (11):

$$\mathbf{\Xi} = \mathbf{\Phi}^{-1}\mathbf{\Theta}. \qquad (12)$$

This time, we utilized Microsoft Excel to obtain the solution.

Table 14 shows the result of computation with GAMS/CONOPT4 in CNS format. Note that the results obtained with other solvers such as CONOPT and MINOS are perfectly identical with those with CONOPT4, unlike the previously seen cases in NLP format (Tables 16 and 17 for reference). In addition, the result provided by Microsoft Excel, which is captured by Table 15, also is identical to the estimated values on Table 14. These facts suggest that solving a system of simultaneous equations composed of FOCs may yield more reliable solution compared to handling the balancing program in NLP format with GAMS. Meanwhile, reliable solutions do not necessarily provide better estimates.

Finally, let us recall Case 1D in NLP format (Table 9, calculated with CONOPT4) and compare it with the similar results obtained by different solvers, CONOPT and MINOS (Tables 16 and 17). As mentioned before, these three results completely differ. These are also different from the values shown in Tables 14 and 15, which are estimated by solving a system of equations. What is the cause of the difference in the estimations obtained? Since it has not yet been identified at this moment, further research is needed.

Concluding Remarks

Using the penalty function that equally evaluate the deviations from the initial values regardless of their magnitude as the objective, we may accidentally encounter one of non-unique solutions. In this case, switching the solver would lead us to a deeper rabbit hole or open a worm can. There is no definitive criterion for deciding which estimate to adopt.

To tackle this problem, there seem to be two solutions: (a) define the penalty function that evaluate the deviations from the initial values with certain weights, and/or (b) formulate the program as a system of simultaneous equations that consists of FOCs, instead of formulating it in the NLP format with GAMS. In these approaches, almost all of the obtained estimates would converge to the same level, regardless of the choice of a solver.

On the other hand, those unique solutions may not necessarily provide good estimates, from the viewpoint of reproducibility of the original data. Furthermore, rich availability of additional information does not necessarily lead to better results, in terms of improving the accuracy of the estimates. However, the present study only covered a limited number of situations, so that the results obtained here cannot be generalized. Thus, further research is needed.

References

Brooke, A., D. Kendrick, and A. Meeraus (1992). *GAMS: A User's Guide*, Scientific Press: San Francisco.

Ministry of Internal Affairs and Communications (2019). *2015 Input-Output Tables for Japan* (https://www.soumu.go.jp/english/dgpp_ss/data/io/io15_00001.htm).

Rutherford, T. F., and A. Schreiber (2019). "Tools for Open Source, Subnational CGE Modeling with an Illustrative Analysis of Carbon Leakage," *Journal of Global Economic Analysis*, 4(2), pp. 1-66.

[1] Another type of interesting objective setting can be found in Rutherford and Schreiber (2019).
[2] Brooke, Kendrick, and Meeraus (1992).

Table 1. Correspondence between Variables and an Input-Output Table

	S01	S02	S03	F01	F02	F03	TTL
S01							
S02		X_{ij}			Z_{if}		CT_i
S03							
V01							
V02		Y_{vj}					
V03							
TTL		CT_j					

Source: Illustration by the author.

Table 2. 2015 Input-Output Table for Japan (Aggregated)

	S01	S02	S03	F01	F02	F03	TTL
S01	1568262	11080682	1976700	4233584	157591	-5470216	13546603
S02	3018302	142567177	73644181	98576226	65612654	-81883812	301534728
S03	2493010	44201065	189030295	460827613	20999173	-14814099	702737057
V01	1645701	45440731	218712786				
V02	5031642	47889990	197110311				
V03	-210314	10355083	22262784				
TTL	13546603	301534728	702737057				

Source: Ministry of Internal Affairs and Communications (2019).

Table 3. 2015 Input-Output Table for Japan (Rounded to the Nearest Ten Thousand)

	S01	S02	S03	F01	F02	F03	TTL
S01	1600000	11100000	2000000	4200000	200000	-5500000	13600000
S02	3000000	142600000	73600000	98600000	65600000	-81900000	301500000
S03	2500000	44200000	189000000	460800000	21000000	-14800000	702700000
V01	1600000	45400000	218700000				
V02	5000000	47900000	197100000				
V03	-200000	10400000	22300000				
TTL	13500000	301600000	702700000				

Source: Manipulations by the author.

Table 4. Deviations from the Original Table (%)

	S01	S02	S03	F01	F02	F03	TTL
S01	2.02376899	0.17433945	1.17873223	-0.79327586	26.91080074	0.54447576	0.39417262
S02	-0.60636742	0.02302283	-0.05999252	0.02411738	-0.01928591	0.01976948	-0.01151708
S03	0.28038395	-0.00240944	-0.01602653	-0.00599205	0.00393825	-0.09517285	-0.00527324
V01	-2.77699290	-0.08963544	-0.00584602				
V02	-0.62886032	0.02090207	-0.00523108				
V03	-4.90409578	0.43376765	0.16716687				
TTL	-0.34401983	0.02164659	-0.00527324				

Source: Calculations by the author.

Table 5. Optimization Problems

		Constraints			
		(3), (4)	(5)	(5), (6), (7)	(3), (4), (6), (7)
Objective Function	Type (1)	Case 1A	Case 1B	Case 1C	Case 1D
	Type (2)	Case 2A	Case 2B	Case 2C	Case 2D

Source: Tabulation by the author.

Table 6. Objective Function Type (1) with Control Totals

	S01	S02	S03	F01	F02	F03	TTL
S01	1609290	11099981	1999986	4199986	199986	-5562626	13546603
S02	3009304	142599995	73600001	98600000	65600000	-81874572	301534728
S03	2509305	44199997	189000002	460800001	21000001	-14772249	702737057
V01	1609304	45399995	218700001				
V02	5009304	47899995	197100001				
V03	-199903	10334765	22337067				
TTL	13546603	301534728	702737057				

	S01	S02	S03	F01	F02	F03	TTL
S01	2.61612671	0.17416875	1.17804922	-0.79360882	26.90185601	1.68933206	0.00000000
S02	-0.29811932	0.02301948	-0.05999166	0.02411742	-0.01928585	-0.01128453	0.00000000
S03	0.65363429	-0.00241729	-0.01602550	-0.00599175	0.00394468	-0.28249880	0.00000000
V01	-2.21165223	-0.08964605	-0.00584575				
V02	-0.44395414	0.02089201	-0.00523078				
V03	-4.95003563	-0.19621475	0.33366325				
TTL	0.00000000	0.00000000	0.00000000				

	S01	S02	S03	F01	F02	F03	TTL
S01	-0.59235772	0.00017070	0.00068301	-0.00033296	0.00894473	-1.14485630	0.39417262
S02	0.30824810	0.00000335	0.00000086	-0.00000004	0.00000006	0.00848495	0.01151708
S03	-0.37325033	-0.00000785	0.00000103	0.00000029	-0.00000643	-0.18732595	0.00527324
V01	0.56534068	-0.00001061	0.00000027				
V02	0.18490618	0.00001006	0.00000030				
V03	-0.04593985	0.23755289	-0.16649638				
TTL	0.34401983	0.02164659	0.00527324				

Source: Calculations by the author.

Table 7. Objective Function Type (1) without Control Totals

	S01	S02	S03	F01	F02	F03	TTL
S01	1600000	11100000	2000000	4200000	200000	-5600000	13500000
S02	3000000	142600000	73600000	98600000	65600000	-81800000	301600000
S03	2500000	44200000	189000000	460800000	21000000	-14800000	702700000
V01	1600000	45400000	218700000				
V02	5000000	47900000	197100000				
V03	-200000	10400000	22300000				
TTL	13500000	301600000	702700000				

	S01	S02	S03	F01	F02	F03	TTL
S01	2.02376899	0.17433945	1.17873223	-0.79327586	26.91080074	2.37255713	-0.34401983
S02	-0.60636742	0.02302283	-0.05999252	0.02411738	-0.01928591	-0.10235479	0.02164659
S03	0.28038395	-0.00240944	-0.01602653	-0.00599205	0.00393825	-0.09517285	-0.00527324
V01	-2.77699290	-0.08963544	-0.00584602				
V02	-0.62886032	0.02090207	-0.00523108				
V03	-4.90409578	0.43376765	0.16716687				
TTL	-0.34401983	0.02164659	-0.00527324				

	S01	S02	S03	F01	F02	F03	TTL
S01	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	-1.82808138	0.05015279
S02	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	-0.08258531	-0.01012951
S03	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
V01	0.00000000	0.00000000	0.00000000				
V02	0.00000000	0.00000000	0.00000000				
V03	0.00000000	0.00000000	0.00000000				
TTL	0.00000000	0.00000000	0.00000000				

Source: Calculations by the author.

Table 8. Objective Function Type (1) without Control Totals (Other Constraints Applied)

	S01	S02	S03	F01	F02	F03	TTL
S01	1600000	11100000	2000000	4200000	200000	-5600000	13500000
S02	3000000	142600000	73600000	98600000	65600000	-81800000	301600000
S03	2500000	44200000	189000000	460800000	20969418	-14730704	702738714
V01	1600000	45400000	218700000				
V02	5000000	47900000	197100000				
V03	-200000	10400000	22338714				
TTL	13500000	301600000	702738714				

	S01	S02	S03	F01	F02	F03	TTL
S01	2.02376899	0.17433945	1.17873223	-0.79327586	26.91080074	2.37255713	-0.34401983
S02	-0.60636742	0.02302283	-0.05999252	0.02411738	-0.01928591	-0.10235479	0.02164659
S03	0.28038395	-0.00240944	-0.01602653	-0.00599205	-0.14169606	-0.56294345	0.00023579
V01	-2.77699290	-0.08963544	-0.00584602				
V02	-0.62886032	0.02090207	-0.00523108				
V03	-4.90409578	0.43376765	0.34106247				
TTL	-0.34401983	0.02164659	0.00023579				

	S01	S02	S03	F01	F02	F03	TTL
S01	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	-1.82808138	0.05015279
S02	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	-0.08258531	-0.01012951
S03	0.00000000	0.00000000	0.00000000	0.00000000	-0.13775781	-0.46777060	0.00503745
V01	0.00000000	0.00000000	0.00000000				
V02	0.00000000	0.00000000	0.00000000				
V03	0.00000000	0.00000000	-0.17389559				
TTL	0.00000000	0.00000000	0.00503745				

Source: Calculations by the author.

Table 9. Objective Function Type (1) with Control Totals and Additional Constraints

	S01	S02	S03	F01	F02	F03	TTL
S01	1600000	11100000	2000000	4200000	200000	-5553397	13546603
S02	3000000	142600000	73600000	98600000	65600000	-81865272	301534728
S03	2546603	44134728	188998343	460800000	20969418	-14712035	702737057
V01	1600000	45400000	218700000				
V02	5000000	47900000	197100000				
V03	-200000	10400000	22338714				
TTL	13546603	301534728	702737057				

	S01	S02	S03	F01	F02	F03	TTL
S01	2.02376899	0.17433945	1.17873223	-0.79327586	26.91080074	1.52061637	0.00000000
S02	-0.60636742	0.02302283	-0.05999252	0.02411738	-0.01928591	-0.02264184	0.00000000
S03	2.14973065	-0.15008009	-0.01690311	-0.00599205	-0.14169606	-0.68896529	0.00000000
V01	-2.77699290	-0.08963544	-0.00584602				
V02	-0.62886032	0.02090207	-0.00523108				
V03	-4.90409578	0.43376765	0.34106247				
TTL	0.00000000	0.00000000	0.00000000				

	S01	S02	S03	F01	F02	F03	TTL
S01	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	-0.97614061	0.39417262
S02	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	-0.00287236	0.01151708
S03	-1.86934669	-0.14767065	-0.00087658	0.00000000	-0.13775781	-0.59379244	0.00527324
V01	0.00000000	0.00000000	0.00000000				
V02	0.00000000	0.00000000	0.00000000				
V03	0.00000000	0.00000000	-0.17389559				
TTL	0.34401983	0.02164659	0.00527324				

Source: Calculations by the author.

Table 10. Objective Function Type (2) with Control Totals

	S01	S02	S03	F01	F02	F03	TTL
S01	1602028	11072152	1995428	4190240	199535	-5512781	13546603
S02	3011080	142588188	73610313	98610077	65606702	-81891631	301534728
S03	2509077	44193577	189014681	460818300	21000834	-14799412	702737057
V01	1605746	45391599	218708298				
V02	5017955	47891136	197107490				
V03	-199282	10398076	22300847				
TTL	13546603	301534728	702737057				

	S01	S02	S03	F01	F02	F03	TTL
S01	2.15305906	-0.07697813	0.94745735	-1.02380919	26.61588958	0.77811772	0.00000000
S02	-0.23928659	0.01473744	-0.04598871	0.03433966	-0.00907100	0.00954934	0.00000000
S03	0.64447545	-0.01694173	-0.00825979	-0.00202085	0.00790974	-0.09914041	0.00000000
V01	-2.42786298	-0.10812270	-0.00205208				
V02	-0.27201641	0.00239398	-0.00143132				
V03	-5.24558723	0.41518395	0.17096942				
TTL	0.00000000	0.00000000	0.00000000				

	S01	S02	S03	F01	F02	F03	TTL
S01	-0.12929008	0.09736132	0.23127488	-0.23053332	0.29491116	-0.23364196	0.39417262
S02	0.36708083	0.00828539	0.01400381	-0.01022228	0.01021491	0.01022013	0.01151708
S03	-0.36409150	-0.01453229	0.00776674	0.00397119	-0.00397149	-0.00396756	0.00527324
V01	0.34912992	-0.01848726	0.00379395				
V02	0.35684391	0.01850809	0.00379976				
V03	-0.34149145	0.01858370	-0.00380255				
TTL	0.34401983	0.02164659	0.00527324				

Source: Calculations by the author.

Table 11. Objective Function Type (2) without Control Totals

	S01	S02	S03	F01	F02	F03	TTL
S01	1600000	11067714	1994432	4188316	199444	-5515301	13534604
S02	3008726	142600000	73609158	98612496	65608313	-81889622	301549072
S03	2506961	44194498	189000000	460801050	21000048	-14799966	702702590
V01	1604451	45394248	218699496				
V02	5013910	47893930	197099555				
V03	-199444	10398682	22299949				
TTL	13534604	301549072	702702590				

	S01	S02	S03	F01	F02	F03	TTL
S01	2.02376899	-0.11703514	0.89702761	-1.06926480	26.55774058	0.82418625	-0.08857667
S02	-0.31726349	0.02302283	-0.04755707	0.03679368	-0.00661563	0.00709485	0.00475687
S03	0.55958764	-0.01485684	-0.01602653	-0.00576418	0.00416582	-0.09540005	-0.00490465
V01	-2.50652279	-0.10229455	-0.00607636				
V02	-0.35241489	0.00822734	-0.00545676				
V03	-5.16864833	0.42104060	0.16693910				
TTL	-0.08857667	0.00475687	-0.00490465				

	S01	S02	S03	F01	F02	F03	TTL
S01	0.00000000	0.05730431	0.28170462	-0.27598894	0.35306016	-0.27971050	0.30559595
S02	0.28910393	0.00000000	0.01243545	-0.01267630	0.01267029	0.01267462	0.00676021
S03	-0.27920369	-0.01244739	0.00000000	0.00022787	-0.00022757	-0.00022720	0.00036859
V01	0.27047011	-0.01265911	-0.00023034				
V02	0.27644543	0.01267474	-0.00022568				
V03	-0.26455255	0.01272705	0.00022777				
TTL	0.25544316	0.01688973	0.00036859				

Source: Calculations by the author.

Table 12. Objective Function Type (2) without Control Totals (Other Constraints Applied)

	S01	S02	S03	F01	F02	F03	TTL
S01	1600000	11067307	1994446	4188752	199350	-5514730	13535125
S02	3008836	142600000	73612388	98626348	65580004	-81878115	301549461
S03	2506943	44192561	189000000	460845577	20990064	-14798536	702736608
V01	1604552	45395444	218714863				
V02	5014225	47895193	197113395				
V03	-199431	10398956	22301516				
TTL	13535125	301549461	702736608				

	S01	S02	S03	F01	F02	F03	TTL
S01	2.02376899	-0.12070905	0.89775507	-1.05896435	26.49831906	0.81374692	-0.08473235
S02	-0.31361838	0.02302283	-0.04317138	0.05084585	-0.04976203	-0.00695783	0.00488597
S03	0.55886636	-0.01924027	-0.01602653	0.00389815	-0.04337684	-0.10505423	-0.00006389
V01	-2.50039344	-0.09966145	0.00094981				
V02	-0.34614942	0.01086497	0.00156480				
V03	-5.17464363	0.42368912	0.17397447				
TTL	-0.08473235	0.00488597	-0.00006389				

	S01	S02	S03	F01	F02	F03	TTL
S01	0.00000000	0.05363040	0.28097716	-0.26568849	0.41248168	-0.26927116	0.30944027
S02	0.29274905	0.00000000	0.01682114	-0.02672847	-0.03047611	0.01281164	0.00663111
S03	-0.27848241	-0.01683083	0.00000000	0.00209389	-0.03943859	-0.00988138	0.00520935
V01	0.27659946	-0.01002601	0.00489621				
V02	0.28271090	0.01003710	0.00366628				
V03	-0.27054785	0.01007853	-0.00680759				
TTL	0.25928748	0.01676062	0.00520935				

Source: Calculations by the author.

Table 13. Objective Function Type (2) with Control Totals and Additional Constraints

	S01	S02	S03	F01	F02	F03	TTL
S01	1601953	11071593	1995331	4190597	199441	-5512313	13546603
S02	3011110	142589093	73610916	98624057	65579568	-81880017	301534728
S03	2508896	44190200	189000582	460845509	20990409	-14798538	702737057
V01	1605799	45392926	218715090				
V02	5018121	47892537	197113599				
V03	-199275	10398380	22301539				
TTL	13546603	301534728	702737057				

	S01	S02	S03	F01	F02	F03	TTL
S01	2.14831219	-0.08202625	0.94254104	-1.01537033	26.55618978	0.76956584	0.00000000
S02	-0.23826885	0.01537232	-0.04516948	0.04852234	-0.05042687	-0.00463509	0.00000000
S03	0.63720176	-0.02458121	-0.01571873	0.00388343	-0.04173383	-0.10503949	0.00000000
V01	-2.42464481	-0.10520273	0.00105331				
V02	-0.26872766	0.00531756	0.00166825				
V03	-5.24873480	0.41811975	0.17407816				
TTL	0.00000000	0.00000000	0.00000000				

	S01	S02	S03	F01	F02	F03	TTL
S01	-0.12454320	0.09231320	0.23619119	-0.22209447	0.35461096	-0.22509009	0.39417262
S02	0.36809857	0.00765051	0.01482304	-0.02440497	-0.03114095	0.01513438	0.01151708
S03	-0.35681780	-0.02217177	0.00030780	0.00210861	-0.03779558	-0.00986664	0.00527324
V01	0.35234809	-0.01556729	0.00479272				
V02	0.36013266	0.01558451	0.00356283				
V03	-0.34463902	0.01564790	-0.00691129				
TTL	0.34401983	0.02164659	0.00527324				

Source: Calculations by the author.

Table 14. Objective Function Type (1) with Control Totals and Additional Constraints (CNS)

	S01	S02	S03	F01	F02	F03	TTL
S01	1601868	11061006	1998746	4194473	199987	-5509478	13546603
S02	3009413	142591990	73614328	98627033	65573312	-81881349	301534728
S03	2506521	44194153	189001648	460838576	20996119	-14799960	702737057
V01	1602671	45394264	218712254				
V02	5026088	47893615	197109953				
V03	-199958	10399699	22300127				
TTL	13546603	301534728	702737057				

	S01	S02	S03	F01	F02	F03	TTL
S01	2.14290166	-0.17757186	1.11530399	-0.92382259	26.90262024	0.71773503	0.00000000
S02	-0.29449968	0.01740478	-0.04053649	0.05154079	-0.05996106	-0.00300821	0.00000000
S03	0.54194082	-0.01563737	-0.01515469	0.00237908	-0.01454375	-0.09544147	0.00000000
V01	-2.61466746	-0.10225780	-0.00024321				
V02	-0.11038530	0.00756992	-0.00018160				
V03	-4.92394253	0.43086104	0.16773916				
TTL	0.00000000	0.00000000	0.00000000				

	S01	S02	S03	F01	F02	F03	TTL
S01	-0.11913268	-0.00323241	0.06342824	-0.13054673	0.00818050	-0.17325928	0.39417262
S02	0.31186774	0.00561806	0.01945603	-0.02742341	-0.04067514	0.01676127	0.01151708
S03	-0.26155686	-0.01322792	0.00087184	0.00361296	-0.01060550	-0.00026862	0.00527324
V01	0.16232544	-0.01262236	0.00560281				
V02	0.51847502	0.01333216	0.00504948				
V03	-0.01984675	0.00290661	-0.00057229				
TTL	0.34401983	0.02164659	0.00527324				

Source: Calculations by the author.

Table 15. Objective Function Type (1) with Control Totals and Additional Constraints (Excel)

	S01	S02	S03	F01	F02	F03	TTL
S01	1601868	11061006	1998746	4194473	199987	-5509478	13546603
S02	3009413	142591990	73614328	98627033	65573312	-81881349	301534728
S03	2506521	44194153	189001648	460838576	20996119	-14799960	702737057
V01	1602671	45394264	218712254				
V02	5026088	47893615	197109953				
V03	-199958	10399699	22300127				
TTL	13546603	301534728	702737057				

	S01	S02	S03	F01	F02	F03	TTL
S01	2.14290166	-0.17757186	1.11530399	-0.92382259	26.90262024	0.71773503	0.00000000
S02	-0.29449968	0.01740478	-0.04053649	0.05154079	-0.05996106	-0.00300821	0.00000000
S03	0.54194082	-0.01563737	-0.01515469	0.00237908	-0.01454375	-0.09544147	0.00000000
V01	-2.61466746	-0.10225780	-0.00024321				
V02	-0.11038530	0.00756992	-0.00018160				
V03	-4.92394253	0.43086104	0.16773916				
TTL	0.00000000	0.00000000	0.00000000				

	S01	S02	S03	F01	F02	F03	TTL
S01	-0.11913268	-0.00323241	0.06342824	-0.13054673	0.00818050	-0.17325928	0.39417262
S02	0.31186774	0.00561806	0.01945603	-0.02742341	-0.04067514	0.01676127	0.01151708
S03	-0.26155686	-0.01322792	0.00087184	0.00361296	-0.01060550	-0.00026862	0.00527324
V01	0.16232544	-0.01262236	0.00560281				
V02	0.51847502	0.01333216	0.00504948				
V03	-0.01984675	0.00290661	-0.00057229				
TTL	0.34401983	0.02164659	0.00527324				

Source: Calculations by the author.

Table 16. Objective Function Type (1) with Control Totals and Additional Constraints (CONOPT)

	S01	S02	S03	F01	F02	F03	TTL
S01	1609435	11034057	2035604	4167531	199975	-5500000	13546603
S02	3000065	142600000	73600149	98649792	65584721	-81900000	301534728
S03	2500065	44200000	189000298	460851972	20984721	-14800000	702737057
V01	1636436	45400224	218700298				
V02	5000298	47900224	197100298				
V03	-199698	10400224	22300410				
TTL	13546603	301534728	702737057				

	S01	S02	S03	F01	F02	F03	TTL
S01	2.62542208	-0.42077527	2.97989911	-1.56020711	26.89511812	0.54447576	0.00000000
S02	-0.60420681	0.02302283	-0.05979011	0.07462892	-0.04257204	0.01976948	0.00000000
S03	0.28299982	-0.00240944	-0.01586882	0.00528599	-0.06882006	-0.09517285	0.00000000
V01	-0.56295347	-0.08914339	-0.00570972				
V02	-0.62293542	0.02136895	-0.00507984				
V03	-5.04792208	0.43592688	0.16900813				
TTL	0.00000000	0.00000000	0.00000000				

	S01	S02	S03	F01	F02	F03	TTL
S01	-0.60165309	-0.24643582	-1.80116688	-0.76693124	0.01568262	0.00000000	0.39417262
S02	0.00216061	0.00000000	0.00020241	-0.05051154	-0.02328612	0.00000000	0.01151708
S03	-0.00261586	0.00000000	0.00015771	0.00070605	-0.06488181	0.00000000	0.00527324
V01	2.21403943	0.00049205	0.00013631				
V02	0.00592490	-0.00046688	0.00015125				
V03	-0.14382630	-0.00215923	-0.00184126				
TTL	0.34401983	0.02164659	0.00527324				

Source: Calculations by the author.

Table 17. Objective Function Type (1) with Control Totals and Additional Constraints (MINOS)

	S01	S02	S03	F01	F02	F03	TTL
S01	1600000	10940305	2037057	4269296	199945	-5500000	13546603
S02	2970832	142694423	73600000	98600000	65569473	-81900000	301534728
S03	2537057	44200000	189000000	460800000	21000000	-14800000	702737057
V01	1638714	45400000	218700000				
V02	5000000	47900000	197100000				
V03	-200000	10400000	22300000				
TTL	13546603	301534728	702737057				

	S01	S02	S03	F01	F02	F03	TTL
S01	2.02376899	-1.26685913	3.05342237	0.84354060	26.87566667	0.54447576	0.00000000
S02	-1.57273858	0.08925310	-0.05999252	0.02411738	-0.06581144	0.01976948	0.00000000
S03	1.76682003	-0.00240944	-0.01602653	-0.00599205	0.00393825	-0.09517285	0.00000000
V01	-0.42456072	-0.08963544	-0.00584602				
V02	-0.62886032	0.02090207	-0.00523108				
V03	-4.90409578	0.43376765	0.16716687				
TTL	0.00000000	0.00000000	0.00000000				

	S01	S02	S03	F01	F02	F03	TTL
S01	0.00000000	-1.09251968	-1.87469014	-0.05026474	0.03513407	0.00000000	0.39417262
S02	-0.96637116	-0.06623027	0.00000000	0.00000000	-0.04652553	0.00000000	0.01151708
S03	-1.48643608	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00527324
V01	2.35243219	0.00000000	0.00000000				
V02	0.00000000	0.00000000	0.00000000				
V03	0.00000000	0.00000000	0.00000000				
TTL	0.34401983	0.02164659	0.00527324				

Source: Calculations by the author.